# Rocking Stiffness of Arbitrarily Shaped Embedded Foundations

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**Abstract:** A simple procedure to compute the static and dynamic rocking stiffness of arbitrarily shaped rigid foundations embedded in a reasonably deep and homogeneous soil deposit is developed. The method is based on an improved understanding of the physics of the problem, substantiated by the results of extensive rigorous parametric studies using the boundary element method, and including analytical and numerical results available in the literature. Analytical expressions and dimensionless graphs are developed, applicable to both saturated and unsaturated soils, a variety of basemat shapes, and a wide range of embedment depths and types of contact between the vertical sidewalls and the surrounding soil. A numerical example illustrates the applicability of the method, while a companion paper (Fotopoulou et al. 1989) presents a simple method for determining the radiation damping of embedded foundations.

## Introduction

Determining the response of embedded foundations to rocking static or dynamic loads is a problem of great interest to geotechnical engineers. Such loading arises when foundations support structures subjected to lateral forces due to wind, water, and earth pressure, and when they are exposed to the dynamic effects of earthquakes, man-induced vibrations, and impact moments. Rocking is usually the most critical mode of oscillation for a foundation, since it is often associated with very small radiation damping, and since it may generate detrimental deformations under the edges of shallow foundations ("humping").

A number of solutions has been reported in the literature (Beredugo and Novak 1972; Day 1977; Dominquez and Roesset 1978; Gazetas et al. 1987; Gazetas and Tassoulas 1987a, 1987b; Haritos and Keer 1980; Johnson et al. 1975; Kausel and Roesset 1975; Pais and Kausel 1985; Wolf 1985) for the static as well as the steady-state harmonic response of fully embedded prismatic foundations. Most of them (for embedded geometries) refer to cylindrical and strip foundations; only Dominiquez and Roesset (1978) and Pais and Kausel (1985) provide rigorous results for the response of rectangular foundations with an aspect ratio equal to two. Therefore, when analyzing foundations of noncircular base shapes, the engineer must approximate the actual mat shape by a circle of the same area moment of inertia. However, the validity of this approximation is questionable, especially with long foun-

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| Shape | L/B | D/B | d/D | V | Symbol |
|---|---|---|---|---|---|
| square | 1 | 1 | 1/2, 3/4, 1, 1 | 0.40, 0.49, 0.33, 0.40 | ⊟, ▨, ▧, ▤ |
| | | 2 | 1/4, 2/3, 3/4, 1 | 0.49, 0.40, 0.49, 0.40 | ◪, ⊟, ◪, ■ |
| circle | 1 | 0.5 | 1 | 0.33 | ○ |
| | | 1 | 1 | 0.33 | ◍ |
| | | 2 | 1 | 0.33 | ● |
| strip | ∞ | 0.5 | 1 | 0.25, 0.33, 0.30 | ▽, ▷, △ |
| | | 1 | 1 | 0.25, 0.33, 0.30 | ▼, ▶, ▲ |
| | | 1.5 | 1 | 0.25 | ▽ |
| | | 2 | 1 | 0.33 | ▶ |
| T-shape | 1 | 1 | 1 | 0.40 | ⊩ |
| | 2 | 1 | 1 | 0.40 | ⊫ |

| Shape | L/B | D/B | d/D | V | Symbol |
|---|---|---|---|---|---|
| rectangle | 2 | 1 | 1/3, 1 | 0.40 | ▯, ▤ |
| | | 2 | 1 | 0.40 | ▬ |
| | 4 | 0.5 | 1 | 0.40 | ◇ |
| | | 1 | 1/3, 2/3, 1 | 0.40 | ◈, ◈, ◆ |
| | | 1.5 | 1 | 0.40 | ◈ |
| | | 2 | 1 | 0.40 | ◆ |
| | 6 | 0.5 | 1 | 0.40 | ⊛ |
| | | 1 | 1 | 0.40, 0.49 | ⊛, ⊛ |
| | | 2 | 1 | 0.40 | ⬢ |
| | 10 | 0.5 | 1 | 0.40 | ◠ |
| | | 1 | 1/3, 2/3, 1 | 0.40 | ◓, ◓, ◓ |

FIG. 1. List of Cases Studied and Symbols for Data Points Used in Figs. 5 and 6

dations. Moreover, most of the aforementioned solutions assume the vertical sidewalls of the foundation and the surrounding soil to be in perfect ("welded") contact. In reality, net tensile stresses cannot be sustained at the soil-foundation interface, and the magnitude of interface shear tractions is limited by Coulomb's friction law. Thus, separation and sliding are likely to occur.

In this paper, an efficient boundary element formulation [see Gazetas and Tassoulas (1987a) and Tassoulas (1987) for details] is used in a comprehensive and systematic parametric study of the rocking response of rigid foundations embedded in an elastic halfspace. Results are presented mainly for rectangular base shapes of aspect ratio $L/B$ up to ten, a wide range of embedment depths $D/B$ up to two, three values of Poisson's ratio, and the frequency range of greatest practical interest ($a_0 \leq 2$). Results are also obtained for foundations with T-shaped basemats. Moreover, in every case, the height $d$ of the vertical sidewalls is varied between the extremes of 0 and $D$, the former representing foundations placed in a trench with no sidewall-backfill contact, and the latter corresponding to the usually studied, but not necessarily more realistic, case of fully embedded foundations with complete sidewall-backfill contact throughout the embedment depth. Fig. 1 lists the cases studied.

The results of this study, as well as available pertinent results from the aforementioned literature, have been systematically organized and cast in the form of simple algebraic formulas and dimensionless graphs. Such information can be used in practice even for cases involving base shapes and types of contact other than those specifically considered in this paper, as illustrated with a numerical example.

This paper, which concerns rocking static/dynamic stiffnesses, and the companion paper (Fotopoulou et al. 1989) which concerns rocking radiation damping of embedded foundations, are a continuation of the work reported in Dobry and Gazetas (1986), Gazetas et al. (1987), and Gazetas and Tassoulas (1987a, 1987b), which studied, respectively, the response in all modes of arbitrarily shaped surface foundations (Dobry and Gazetas 1986), the vertical response of arbitrarily shaped embedded foundations (Gazetas et al. 1987), and the horizontal (swaying) stiffness and damping of arbitrarily shaped embedded foundations (Gazetas and Tassoulas 1987a, 1987b).

## Problem Statement

A massless rigid foundation consists of a horizontal basemat of arbitrary solid shape located at a depth $D$ below the ground surface and of a vertical sidewall which may enjoy any degree of contact with the surrounding soil (Fig. 2). The foundation is subjected to steady-state vibration by a harmonic moment about the $x$- or $y$-axis, $M = M_0 \exp(i\omega t)$, having amplitude $M_0$ and circular frequency $\omega$. Static loading is treated as the special case of $\omega = 0$. We seek to estimate the steady-state harmonic rotation $\theta = \theta_0 \exp(i\omega t)$, about the $x$- or $y$-axis, respectively. The horizontal displacement that will inevitably arise upon horizontal loading, being usually (but not always) secondary, is not studied in this paper.

The soil is assumed to be a homogeneous elastic halfspace—a choice stemming from the need to keep the number of independent problem parameters to a minimum, while trying to understand and quantify the role of partial embedment and of basemat shape.

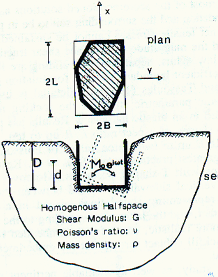



**FIG. 2. Problem Geometry and Loading**

Due to the presence of radiation damping in the system, $M$ is generally out of phase with $\theta$. It has become traditional to introduce complex notation and to express the response through the dynamic moment-rotation ratio

$$\bar{K}_r + i\omega C_r = \frac{M}{\theta} \quad \text{(1)}$$

$\bar{K}_r$ (i.e., $\bar{K}_{rx}$ or $\bar{K}_{ry}$) = the rocking dynamic stiffness; and $C_r$ (i.e., $C_{rx}$ or $C_{ry}$) = the rocking radiation damping coefficient of the soil-foundation system (with respect to the $x$- or $y$-axis, respectively). $\bar{K}_r$ and $C_r$ are both functions of the frequency $\omega$. It is convenient to express $\bar{K}_r(\omega)$ as the product of the static stiffness $K_r$ times a dynamic stiffness coefficient $k_r(\omega)$ as follows:

$$\bar{K}_r(\omega) = K_r \cdot k_r(\omega) \quad \text{(2)}$$

This paper presents information about $K_r$ (i.e., $K_{rx}$ and $K_{ry}$) and $k_r(\omega)$ [i.e., $k_{rx}(\omega)$ and $k_{ry}(\omega)$], while the companion paper develops a simple physical method for $C_r(\omega)$ [i.e., $C_{rx}(\omega)$ and $C_{ry}(\omega)$].

## Static Stiffness of Surface Foundations

For a surface foundation of arbitrary solid shape, the rocking static stiffness $K_{ry,sur}$ for moment about the lateral axis $y$ is always greater than the stiffness $K_{rx,sur}$ for moment about the longitudinal axis $x$. For foundations of different shapes, Dobry and Gazetas (1986) have found that the dimensionless parameters

$$S_{rx} = \frac{(1-\nu)\left(\dfrac{B}{L}\right)^{0.25} K_{rx,sur}}{GI_{bx}^{0.75}} \quad \text{(3a)}$$

$$S_{ry} = \frac{(1 - \nu)K_{ry,sur}}{GI_{by}^{0.75}} \quad (3b)$$

can be uniquely related to the foundation aspect ratio $L/B$, where $2L$ and $2B$ = the length and width of the rectangle circumscribed to the basemat surface, as shown in Fig. 2 ($L > B$); $I_{bx}$ and $I_{by}$ = the area moment of inertia of the basemat; and $G$ and $\nu$ = the shear modulus and Poisson's ratio of the halfspace, respectively. Note that the presented results are *not* sensitive to the exact circumscribed rectangle.

The dependence of $S_{rx}$ and $S_{ry}$ on $B/L$ is shown in Fig. 3. The data points of Dobry and Gazetas (1986) (obtained from published rigorous solutions by several researchers) have been augmented in Fig. 3 with the boundary element results of the present study. The "equivalent circle" approximation predicts $S_{rx} = 3.2B/L$ and $S_{ry} = 3.2$; their agreement with the numerical data, which is very good for the square, the circle, and the short rectangles, deteriorates for longer shapes. On the other hand, the following equally simple expressions fit the data points reasonably well for $0 < B/L < 1$:

$$S_{rx} = 2.4 + 0.5\frac{B}{L} \quad (4a)$$

$$S_{ry} = 3\left(\frac{L}{B}\right)^{0.15} \quad (4b)$$

Thus, combining Eqs. 3 and 4 leads to the following expressions for the static rotational stiffnesses of a surface foundation:

$$K_{rx,sur} = \frac{G}{1 - \nu} I_{bx}^{0.75}\left(\frac{L}{B}\right)^{0.25}\left(2.4 + 0.5\frac{B}{L}\right) \quad (5a)$$

$$K_{ry,sur} = \frac{3G}{1 - \nu} I_{by}^{0.75}\left(\frac{L}{B}\right)^{0.15} \quad (5b)$$

## Effects of Embedment on Static Stiffness

There are three possible effects of embedment on the rocking static stiffness of a rigid foundation. First, in practice, a foundation mat placed at a depth $D$ below the ground surface transmits the load to a soil that is deeper and, therefore, different than the soil affected by a similar surface foundation. Thus, other things being equal, a greater $D$ implies, in general, a somewhat different stiffness. This important effect of embedment is not further addressed herein. However, when applying the proposed method to a practical situation, the engineer must first ensure that the soil deposit is indeed reasonably homogeneous and relatively deep and, second, establish a representative value of soil modulus for the particular depth of embedment.

The other two effects that modify the behavior of embedded foundations and are addressed in this paper are referred to as the "trench" effect and the "sidewall-contact" effect, using the terminology introduced in Gazetas et al. (1987). These two effects are explained with the aid of Fig. 4.

The trench effect stems from the fact that even in a perfectly homogeneous halfspace, the rotation of a foundation placed at the bottom of an open trench

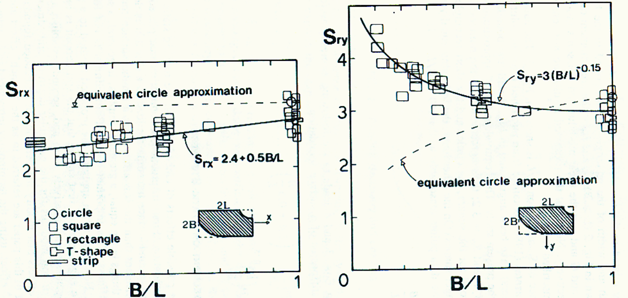



FIG. 3. Surface Foundations of Arbitrary Shape: Rocking Static Stiffness Factors, $S_{rx}$ and $S_{ry}$, versus Ratio $B/L$; [The Scatter of "Rigorous" Data Points Arises Because of Different Values of Poisson's Ratio, Different Interface Conditions ("Rough" versus "Relaxed" Boundaries), and Different Solution Techniques]

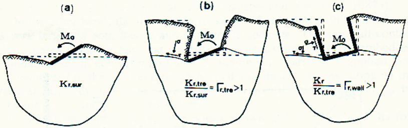



**FIG. 4. Schematic Illustration of Effects of Embedment on Rocking Static Stiffness: (*a*) Surface Foundation; (*b*) Foundation in Trench; (*c*) Fully Embedded Foundation**

is smaller than that of the same foundation on the ground surface. To understand why, visualize a horizontal plane surface passing through the base. In the case of the surface foundation, this plane deforms free of any external stress, while for the embedded foundation, normal and shear stresses from the overlying soil seem to restrict its movement (Fig. 4), thereby increasing the foundation stiffnesses from $K_{rx,sur}$ to $K_{rx,tre}$ and from $K_{ry,sur}$ to $K_{ry,tre}$. One could write $K_{rx,tre} = K_{rx,sur}\Gamma_{rx,tre}$ and $K_{ry,tre} = K_{ry,sur}\Gamma_{ry,sur}$.

However, careful examination of all the data shows that the trench effect is rather insignificant in the case of rocking and $\Gamma_{rx,tre} = \Gamma_{ry,tre}$ may be assumed equal to one for practical applications. In addition to the numerical data, Erden's (1984) experimental work also supports this outcome. By contrast, the trench effect had been found to be appreciable for vertical and

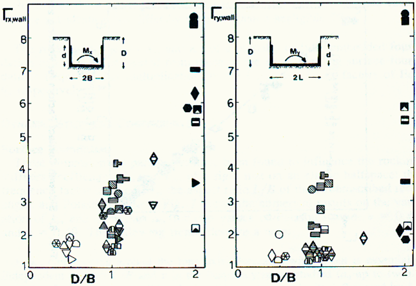



**FIG. 5. Compilation of Numerical Data for Sidewall Contact Effect on Rocking Stiffness about $x$ (Left) or $y$ (Right) Axis**

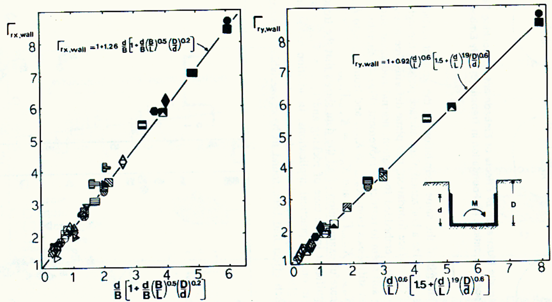



FIG. 6. Sidewall Contact Factors for Rocking about $x$ and $y$ versus Two Dimensionless Geometric Parameters

horizontal loading (Gazetas et al. 1987; Gazetas and Tassoulas 1987a).

On the other hand, the increase in rocking stiffness due to full embedment is very large, as both theory and experiment indicate. The sidewall-contact effect arises from the fact that, when the vertical sidewalls of an embedded foundation are in contact with the surrounding soil, part of the applied load is transmitted to the ground through normal and shear tractions acting on the vertical sides, thereby increasing the foundation stiffness about $x$ and $y$ by a factor $\Gamma_{rx,wall}$ and $\Gamma_{ry,wall}$, respectively.

Careful examination of the data points shown in Fig. 5 leads to the following conclusions:

1. Both factors, $\Gamma_{rx,wall}$ and $\Gamma_{ry,wall}$ may reach very high values, depending on the relative depth of embedment and foundation geometry.

2. Poisson's ratio $\nu$ has no measurable effect on the value of either of the $\Gamma$-factors.

3. For a given basemat, both $\Gamma_{rx,wall}$ and $\Gamma_{ry,wall}$ depend on $d$, $D$, $B$ and $L$ in a complicated way. By trial and error, it is found that an excellent correlation exists between each of them and the dimensionless parameters $L/B$, $D/B$, and $d/D$. Indeed, the scatter of data points in Fig. 6 around the curves

$$\Gamma_{rx,wall} \equiv \frac{K_{rx}}{K_{rx,sur}} \approx 1 + 1.26\frac{d}{B}\left[1 + \frac{d}{B}\left(\frac{D}{d}\right)^{0.2}\left(\frac{B}{L}\right)^{0.5}\right] \quad (6a)$$

$$\Gamma_{ry,wall} \equiv \frac{K_{ry}}{K_{rx,sur}} \approx 1 + 0.92\left(\frac{d}{L}\right)^{0.6}\left[1.5 + \left(\frac{d}{L}\right)^{1.9}\left(\frac{D}{d}\right)^{0.6}\right] \quad (6b)$$

is negligibly small (less than 10%) and independent of basemat shape.

Notice that these two expressions are best fits, which for $L = B$ yield values that are not identical but differ by not more than a negligible 5%.

The rocking stiffnesses $K_{rx}$ and $K_{ry}$ of a partially or fully embedded foundation can thus be computed from those of the corresponding surface foundations (Eqs. 5) after multiplication by the $\Gamma_{rx,wall}$ and $\Gamma_{ry,wall}$ factors of Eq. 6, respectively.

## Dynamic Stiffness Coefficients

### Surface Foundation

Three dimensionless parameters have been found to influence the rocking stiffness coefficients, $k_{rx}$ and $k_{ry}$, of a rigid mat on an elastic halfspace: the frequency factor $a_0 = \omega B/V_s$; the aspect ratio $L/B$ of the circumscribed rectangle; and Poisson's ratio $\nu$. Fig. 7 shows the numerical results on the variation of $k_{rx}$ and $k_{ry}$ versus $a_0$, for two values of Poisson's ratio, $\nu = 0.40$ and $\nu = 0.49$. The following trends deserve a note:

1. $k_{rx}$ (for rocking around the long axis) decreases with $a_0$ but is essentially independent of both $L/B$ and $\nu$ in the frequency range studied ($a_0$ up to two).

2. $k_{ry}$ (for rocking around the short axis) is only marginally influenced by $L/B$ and $\nu$ with one exception: for very long and narrow footings, $k_{ry}$ is sensitive to variations in $\nu$, and when $\nu \approx 0.50$ (typical of saturated clays), the decrease

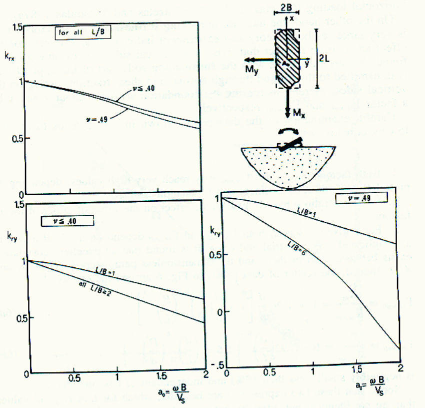



**FIG. 7. Surface Foundations: Proposed Curves for Rocking Dynamic Stiffness Coefficients, $k_{rx}$ and $k_{ry}$, versus Frequency**

of $k_{ry}$ with frequency can be rather dramatic for $L/B \geq 4$.

3. The results for the aforementioned two T-shaped foundations (with aspect ratios of the respective circumscribed rectangles $L/B = 1$ and 2), as well as those for the circle ($L/B = 1$), practically coincide with the curves of Fig. 7 for the corresponding value of $L/B$. This shows that the exact shape of foundation mats that have the same $L/B$ does not have any appreciable effect on $k_{rx}$ or $k_{ry}$ (additional evidence will be given in the sequel).

**Partially and Fully Embedded Foundation**

All the numerical data indicate that the depth and degree of embedment have no substantial effect on either $k_{rx}$ or $k_{ry}$. Some typical results for fully embedded foundations are shown in Fig. 8 for rectangles, and in Figs. 9 and 10 for the T-shaped foundations with $L/B = 1$ and 2 and for a circle. It is evident that use of the pertinent curves proposed for surface foundations in Fig. 7 would provide sufficient accuracy for embedment depth ratios up to at least two. This conclusion is also valid for foundations placed in an open trench, as well as for partially embedded foundations with $d/D$ between zero and one.

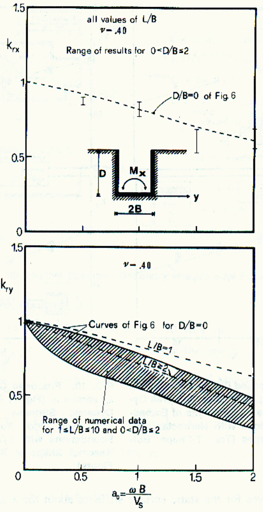



**FIG. 8. Rocking Stiffness Coefficient of Embedded Foundation (Range of Results) Can Be Approximately Taken as Equal to that of Corresponding Surface Foundation (Dashed Lines)**

It is worth reemphasizing the significant message in Fig. 9: an arbitrarily shaped embedded foundation has about the same $k_{rx}$ and $k_{ry}$ as the circumscribed rectangular surface foundation.

## Numerical Example—Comparison with Equivalent Circle Approximation

The developed method is applied to obtain estimates of the dynamic stiffnesses $\bar{K}_{rx}$ and $\bar{K}_{ry}$ for the embedded foundation shown in Fig. 11.

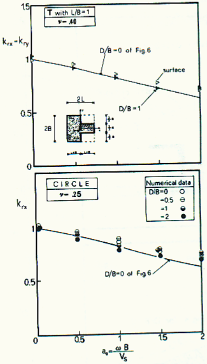



**FIG. 9. Proposed Curves for $k_{rx}$ and $k_{ry}$ versus $a_o$ (Fig. 7) Predict Very Well Dynamic Stiffness Coefficients of Embedded Foundations with Nonrectangular Basemat Shapes (Top: T-Shape; Bottom: Circle)**

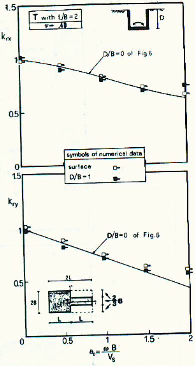



**FIG. 10. Proposed Curves for $k_{rx}$ and $k_{ry}$ versus $a_o$ (Fig. 7) Predict Very Well Dynamic Stiffness Coefficients of Embedded and Surface T-Shaped Foundations with $L/B = 2$ (Detailed Basemat Shape Is Shown in Bottom Figure)**

Eq. 5*a* gives for the static surface stiffness about the *x*-axis

$$K_{rx,sur} \approx \frac{74}{1-0.38} \times (20341)^{0.75} \times (2)^{0.25} \times \left(2.4 + 0.5 \times \frac{1}{2}\right)$$

$$\approx 6.4 \times 10^5 \text{ MN} \cdot \text{m} \quad (7)$$

The sidewall effect is determined from Eq. 6*a*

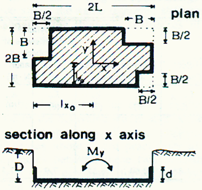
$$\Gamma_{rx,wall} \approx 1 + 1.26 \times \frac{6.6}{10} \times \left[1 + \frac{6.6}{10} \times \left(\frac{11}{6.6}\right)^{0.2} \times \left(\frac{1}{2}\right)^{0.5}\right] \approx 2.26 \quad (8)$$

and thus the static embedded stiffness about *x* is

$$K_{rx} = K_{rx,sur}\Gamma_{rx,wall} = 6.4 \times 10^5 \times 2.26 \approx 14.5 \times 10^5 \text{ MN} \cdot \text{m} \quad (9)$$

From Fig. 7(*a*), the dynamic stiffness coefficient for $a_0 = 1.5$ is

$$k_{rx} \approx 0.75 \quad (10)$$




**Foundation Geometric Parameters**

L = 20 m, B = 10 m

B = 11 m, d = 6.6 m

$I_x$ = 20.341 $m^4$

$I_y$ = 71.700 $m^4$

$\ell_{y_o}$ = 19.54 m, $\ell_{y_o}$ = 9.35 m

**Soil Parameters**

G = 74 MPa

ρ = 1.85 $Mg/m^3$

ν = 0.38

$V_s$ = 200 m/s

**FIG. 11. Illustrative Example: Geometry and Material Parameters**

leading to a dynamic stiffness

$$\bar{K}_{rx} = K_{rx}\cdot k_{rx} = 14.5\times10^5\times0.75 \approx 10.9\times10^5 \text{ MN}\cdot\text{m} \quad (11)$$

Similarly, one obtains for the stiffness about the $y$-axis:

$$K_{ry,sur} \approx \frac{3\times74}{1-0.38}\times(71{,}730)^{0.75}\times(2)^{0.15} \approx 17.4\times10^5 \text{ MN}\cdot\text{m} \quad (12)$$

$$\Gamma_{ry,wall} \approx 1 + 0.92\times\left(\frac{6.6}{20}\right)^{0.6}\times\left[1.5+\left(\frac{6.6}{20}\right)^{1.9}\times\left(\frac{6.6}{11}\right)^{-0.6}\right] \approx 1.79 \quad (13)$$

$$K_{ry} = 17.4\times10^5\times1.79 \approx 31\times10^5 \text{ MN}\cdot\text{m} \quad (14)$$

$$k_{ry} \approx 0.57 \quad (15)$$

$$\bar{K}_{ry} = 31\times10^5\times0.57 \simeq 17.6 \text{ MN}\cdot\text{m} \quad (16)$$

**Equivalent Circle Approximation**

Equating respective area-moments of inertia, one obtains the radii of the equivalent circles for rocking about $x$ and $y$:

$$R_x = \left(\frac{4I_{bx}}{\pi}\right)^{1/4} \approx 12.68 \text{ m} \quad (17)$$

$$R_y = \left(\frac{4I_{by}}{\pi}\right)^{1/4} \approx 17.38 \text{ m} \quad (18)$$

The static stiffnesses of the respective surface foundations are (Dobry and Gazetas 1986; Veletsos and Wei 1971)

$$K^0_{rx,sur} = \frac{8GR_x^3}{3(1-\nu)} = \frac{8\times74\times12.68^3}{3\times(1-0.38)} \approx 6.5\times10^5 \text{ MN}\cdot\text{m} \quad (19)$$

**TABLE 1. Summary of Results of Numerical Example—Comparison with Equivalent Circle**

| Method of analysis (1) | Surface Static Stiffnesses ($10^5$ MN · m) | | Sidewall-Soil Contact Factors | | Dynamic Stiffness Coefficients | | Dynamic Stiffnesses of Embedded Foundation ($10^5$ MN · m) | |
|---|---|---|---|---|---|---|---|---|
| | $K_{rx,sur}$ (2) | $K_{ry,sur}$ (3) | $\Gamma_{rx,wall}$ (4) | $\Gamma_{ry,wall}$ (5) | $k_{rx}$ (6) | $k_{ry}$ (7) | $\bar{K}_{rx}$ (8) | $\bar{K}_{ry}$ (9) |
| Developed in this paper | 6.4 | 17.4 | 2.26 | 1.79 | 0.75 | 0.57 | 10.9 | 17.6 |
| "Equivalent-circle" approximation | 6.5 | 17.0 | 2.03 | 1.88 | 0.60 | 0.55 | 7.9 | 17.5 |

$$K^0_{ry,sur} = \frac{8GR_y^3}{3(1-\nu)} = \frac{8 \times 74 \times 17.38^3}{3 \times (1 - 0.38)} \approx 17 \times 10^5 \text{ MN} \cdot \text{m} \quad (20)$$

The sidewall contact factors, $\Gamma^0_{rx,wall}$ and $\Gamma^0_{ry,wall}$, are estimated from Eqs. 6

$$\Gamma^0_{rx,wall} = 1 + 1.26 \times \frac{6.6}{12.68} \times \left[1 + \frac{6.6}{12.68} \times \left(\frac{11}{6.6}\right)^{0.2} \times 1^{0.5}\right] \approx 2.03 \quad (21)$$

$$\Gamma^0_{ry,wall} = 1 + 0.92 \times \left(\frac{6.6}{17.38}\right)^{0.6} \times \left[1.5 + \left(\frac{6.6}{17.38}\right)^{1.9} \times \left(\frac{11}{6.6}\right)^{0.6}\right] \approx 1.88 \quad (22)$$

Thus, the static stiffnesses of the embedded "equivalent" circular foundations are

$$K^0_{rx} = 6.5 \times 10^5 \times 2.03 \approx 13.2 \times 10^5 \text{ MN} \cdot \text{m} \quad (23)$$

$$K^0_{ry} = 17 \times 10^5 \times 1.88 \approx 32 \times 10^5 \text{ MN} \cdot \text{m} \quad (24)$$

The dimensionless frequency factors of the two "equivalent" circles are

$$a_{0x} = \frac{\omega R_x}{V_s} = \frac{30 \times 12.68}{200} \simeq 1.9 \quad (25)$$

$$a_{0y} = \frac{\omega R_y}{V_s} = \frac{30 \times 17.38}{200} \simeq 2.6 \quad (26)$$

For the $a_{0x}$ value, Fig. 7($a$) gives the dynamic stiffness coefficient

$$k^0_{rx} \approx 0.6 \quad (27)$$

The value of $a_{0y}$, 2.6, is beyond the frequency range covered in Fig. 7, and one must resort to the published solution for the circular foundation (Dobry and Gazetas 1986; Veletsos and Wei 1971) to obtain

$$k^0_{ry} \approx 0.55 \quad (28)$$

(invoking our finding that, in rocking, a surface and an embedded foundation have about the same dynamic stiffness coefficient).

Finally, the dynamic stiffnesses of the "equivalent" circular embedded foundation are

$$\bar{K}^0_{rx} = 13.2 \times 10^5 \times 0.6 \approx 7.9 \times 10^5 \text{ MN}\cdot\text{m} \quad (29)$$

$$\bar{K}^0_{ry} = 32 \times 10^5 \times 0.55 \approx 17.6 \times 10^5 \text{ MN}\cdot\text{m} \quad (30)$$

Table 1 compares the results of the method developed in this paper with those of the equivalent-circle approximation. The largest discrepancy, of about 30%, pertains to the dynamic stiffnesses about the $x$-axis. By contrast, the stiffnesses about $y$ are practically identical. Note, however, that the discrepancies between these two procedures tend to increase with increasing aspect ratio, especially for soils whose Poisson's ratio approaches 0.50 (saturated fine soils).

## Conclusion

Using numerical data obtained with a boundary element formulation, the paper has developed closed-form algebraic expressions and dimensionless charts for estimating inexpensively and reliably the static and dynamic rocking stiffnesses of rigid foundations embedded in a homogeneous halfspace. These results may apply to foundations having an arbitrary (but solid) basemat shape and vertical sidewalls that may only partially (i.e., over a limited height) be effective in transmitting loads into the surrounding soil.

## Appendix. References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